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The purpose of this thesis is to unify several of the definitions of the integral in the setting of convergence theory for nets by showing some of the relations among the definitions. We consider the standard Riemann and Lebesgue integrals and also the relatively new integrals by Edward J. McShane and Ralph Henstock. We prove the equivalence of McShane's integral and Henstock's Riemann-complete integral, and that they each include the Lebesgue integral.

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A CONVERGENCE THEORY APPROACH

TO DEFINITIONS OF

THE INTEGRAL

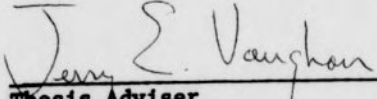
by

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CHAPTER I

INTRODUCTION

As long ago as 200 B.C., the idea of the integral was known to Archimedes. More than two thousand years later, 1665, Sir Isaac Newton and Gottfried Leibniz simultaneously and independently invented the differential and integral calculus. It was almost another two hundred years before Bernhard Riemann [10] gave the first rigorous definition of the integral. G. Darboux [1] and S. Pollard [9] followed quickly with variations of their own. In 1901, in a very short article for Comptes Rendus [6], Henri Lebesgue gave his definition of the integral. Lebesgue's more general definition of the integral requires the use of measure theory. This is not the case, however, for the integrals recently defined by Edward J. McShane [7], and Ralph Henstock [3], [4]. Their integrals are as general as Lebesgue's but do not require the use of measure theory.

The following definition is a polished version of Riemann's definition. We first need some terminology.

Definition 1: A partition of $[a,b]$ is a finite set of points $\{x_i\}_{i=0}^n$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Then we say $P = \{x_i\}_{i=0}^n$ is a partition. At times it is more convenient to refer to P as the set of intervals determined by the points $\{x_i\}_{i=0}^n$. Then $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ is called a partition of $[a,b]$.

Definition 2: The norm of a partition $P = \{x_i\}_{i=0}^n$, denoted $\text{norm } P$, is $\max \{(x_1 - x_0), (x_2 - x_1), \dots, (x_n - x_{n-1})\}$.

Definition 3: Let $P = \{x_i\}_{i=0}^n$ be a partition of $[a, b]$ and let $x_{i-1} \leq \xi_i \leq x_i$ for $i = 1, \dots, n$. Then

$$S(P, f) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$$

is a Riemann sum.

Definition 4: The number I is the Riemann integral of f over $[a, b]$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $\text{norm } P < \delta$ implies $|S(P, f) - I| < \epsilon$ for every Riemann sum $S(P, f)$. Denote this by $\int_a^b f(x) dx$.

For bounded, real-valued functions defined on a closed interval, Riemann's integral is equivalent to Darboux's integral. For this reason we restrict our discussion to those functions which are defined and bounded on a closed interval. All integrals are taken over the interval $[a, b]$. Darboux's own definition of the integral can be found in [1] and is restated here.

Definition 5: Let $P = \{x_i\}_{i=0}^n$ be a partition of $[a, b]$. Then

$$U(P, f) = \sum_{i=1}^n M_i (x_i - x_{i-1}),$$

where $M_i = \sup \{f(x) \mid x_{i-1} \leq x \leq x_i\}$, is called an upper Darboux sum.

Similarly

$$L(P, f) = \sum_{i=1}^n m_i (x_i - x_{i-1}),$$

where $m_i = \inf \{f(x) \mid x_{i-1} \leq x \leq x_i\}$, is called a lower Darboux sum.

Definition 6: The number I is the upper Darboux integral of f , denoted $\bar{\int}_a^b f(x) dx$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\text{norm } P < \delta$ implies $|U(P, f) - I| < \epsilon$. Likewise I is the lower

Darboux integral of f , denoted $\underline{D}f(x)dx$, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\text{norm } P < \delta$ implies $|L(P, f) - I| < \epsilon$.

Definition 7: The number I is the Darboux integral of f , $\underline{D}f(x)dx$, if $\overline{D}f(x)dx = \underline{D}f(x)dx = I$.

In [9], Pollard states his definition of the definite integral. Although different from the Darboux and Riemann integrals in its limiting process, the Pollard integral is equivalent to both the Darboux and Riemann integrals. Again another definition precedes that of the integral.

Definition 8: A partition $P' = \{y_i\}_{i=0}^m$ is a refinement of the partition $P = \{x_i\}_{i=0}^n$, both partitions of $[a, b]$, if $x_i \in P$ implies $x_i \in P'$ for $i = 1, \dots, n$.

Definition 9: The number I is the Pollard integral of f if for every $\epsilon > 0$ there is a partition P such that P' a refinement of P implies that $|S(P', f) - I| < \epsilon$ for every Riemann sum $S(P', f)$. This integral is denoted $Pf(x)dx$.

Most elementary calculus books in use today give the following definition for the definite integral and incorrectly call it the Riemann integral. However, it is equivalent to the Riemann integral. Since it is a combination of the ideas of Darboux and Pollard we call it the Darboux-Pollard integral.

Definition 10: The number I is the upper Darboux-Pollard integral of f , $\overline{DP}f(x)dx$, if for every $\epsilon > 0$ there is a partition P such that P' a refinement of P implies that $|U(P', f) - I| < \epsilon$. Similarly I is the lower Darboux-Pollard integral of f , denoted

$\underline{DP}\int f(x)dx$, if for every $\epsilon > 0$ there is a partition P such that
 $|L(P', f) - I| < \epsilon$ for every P' finer than P .

Definition 11: The number I is the Darboux-Pollard integral of f , denoted $\underline{DP}\int f(x)dx$, if $\overline{DP}\int f(x)dx = \underline{DP}\int f(x)dx = I$.

These four integrals, $R\int f(x)dx$, $D\int f(x)dx$, $P\int f(x)dx$ and $\underline{DP}\int f(x)dx$, are redefined and proved equivalent in the setting of convergence theory for nets in Chapter III. This is done using the tools of convergence theory developed in Chapter II.

In Chapter IV we present the standard definitions from Lebesgue integration theory and restate the definition of the Lebesgue integral in terms of convergence of nets.

Two relatively new definitions for the definite integral are stated, discussed and redefined in Chapter V. That these, the McShane integral [7] and the Riemann-complete integral of Henstock [4] are equivalent, is proved in Chapter V.

We now present the essential ideas from convergence theory for nets.

CHAPTER II

CONVERGENCE THEORY OF NETS

The following definitions are basic to convergence theory. The first five are from a book by John L. Kelley [5]. The definition of an approximate subnet (Definition 17) is due to B. J. Pettis [8]. The definitions are stated for the case when the range of the net is a metric space. In fact, for each net defined in this thesis the range is the real numbers.

Definition 12: A set D is directed by the binary relation \leq if D is non-empty and

- i) if m, n and p are in D with $n \leq m$
and $p \leq n$ then $p \leq m$;
- ii) if $m \in D$, then $m \leq m$; and
- iii) if m and n are in D , then there is a $p \in D$ such that
 $m \leq p$ and $n \leq p$.

Then we say that \leq directs D or that \leq is a direction in D .

Definition 13: A directed set is an ordered pair (D, \leq) , where \leq directs D .

Definition 14: A net is a function defined on a directed set.

Definition 15: A net μ defined on (D, \leq) into X , denoted $\mu: (D, \leq) \rightarrow X$, converges to a point $p \in X$ if for every $\varepsilon > 0$ there is a $d \in D$ such that $\{\mu(d') \mid d' \geq d\} \subset N_\varepsilon(p)$, the neighborhood about p of radius ε .

Definition 16: Let $\mu: (D, \leq) \rightarrow X$ and $\nu: (E, <) \rightarrow X$ each be nets. Then ν is a subnet of μ if there exists a net, $N: (E, <) \rightarrow (D, \leq)$, such that

- i) $\nu = \mu \circ N$; and
- ii) for every $d \in D$ there is an $e \in E$ such that $\{N(e') \mid e' > e\} \subset \{d' \mid d' \geq d\}$.

Definition 17: Let $\mu: (D, \leq) \rightarrow X$ and $\nu: (E, <) \rightarrow X$ each be nets. Then ν is an approximate subnet of μ if for every $\varepsilon > 0$ and for every $d \in D$ there is an $e \in E$ such that $\{\nu(e') \mid e' > e\} \subset N_\varepsilon(\{\mu(d') \mid d' \geq d\})$.

Lemma 1: [5] Let $\mu: (D, \leq) \rightarrow X$ be a net. If μ converges to a point $p \in X$, then so does every subnet of μ .

Proof: Let $\nu: (E, <) \rightarrow X$ be a subnet of μ . Let $\varepsilon > 0$. Since μ converges to p , there is a $d \in D$ such that $\{\mu(d') \mid d' \geq d\} \subset N_\varepsilon(p)$. Since ν is a subnet of μ there is an $e \in E$ such that $\{\nu(e') \mid e' > e\} \subset \{\mu(d') \mid d' \geq d\} \subset N_\varepsilon(p)$. This implies that ν converges to p .

Lemma 2: [8] Let $\mu: (D, \leq) \rightarrow X$ be a net. If μ converges to a point $p \in X$ then so does every approximate subnet of μ .

Proof: Let $\nu: (E, <) \rightarrow X$ be an approximate subnet of μ . Let $\varepsilon > 0$. Since μ converges to p , there is a $d \in D$ such that $\{\mu(d') \mid d' \geq d\} \subset N_{\frac{\varepsilon}{2}}(p)$. Since ν is an approximate subnet of μ , there is an $e \in E$ such that $\{\nu(e') \mid e' > e\} \subset N_{\frac{\varepsilon}{2}}(\{\mu(d') \mid d' \geq d\})$. This implies $\{\nu(e') \mid e' > e\} \subset N_\varepsilon(p)$. Therefore ν converges to p .

Lemma 3: Let $\mu: (D, \leq) \rightarrow X$ and $\nu: (E, <) \rightarrow X$ be nets. If there exists a net, $N: (E, <) \rightarrow (D, \leq)$, such that $\nu = \mu \circ N$, $N(E)$ is cofinal in D and N is order preserving, that is $e' > e$ implies $N(e') \geq N(e)$, then ν is a subnet of μ .

Proof: Let $d \in D$. Since $N(E)$ is cofinal in D , there is an $e \in E$ such that $N(e) \geq d$. Let $e' > e$, then $N(e') \geq N(e)$ and so $\{N(e') \mid e' > e\} \subset \{d' \mid d' \geq d\}$. Therefore ν is a subnet of μ .

Definition 18: A net $\mu: (D, <) \rightarrow R$ is monotonically increasing if $d' > d$ implies $\mu(d') \geq \mu(d)$ and μ is monotonically decreasing if $d' > d$ implies $\mu(d') \leq \mu(d)$.

Lemma 4: Let $\mu: (D, <) \rightarrow R$ be a monotonically decreasing net which is bounded below. Then μ converges.

Proof: Let $\ell = \inf \{\mu(d) \mid d \in D\}$, then μ converges to ℓ . Since μ is bounded below, $\ell \in R$. For every $\epsilon > 0$, there is a $d \in D$ such that $\mu(d) < \ell + \epsilon$. For all $d \in D$, $\mu(d) \geq \ell$. Let $d' > d$, then $\mu(d') \leq \mu(d)$ and so $\mu(d') < \ell + \epsilon$. Therefore μ converges to ℓ .

Similarly if $\mu: (D, <) \rightarrow R$ is a monotonically increasing net which is bounded above then μ converges.

CHAPTER III

THE RIEMANN TYPE INTEGRALS

Each of the definitions of the integral as stated in Chapter I, has a natural redefinition in convergence theory of nets. First set

$$\mathcal{P} = \{P \mid P \text{ is a finite partition of } [a,b]\} \text{ and}$$

$$\mathcal{D} = \{d = (P, \{\xi_i\}_{i=1}^n) \mid P \in \mathcal{P}, P = \{x_i\}_{i=1}^n \text{ and } x_{i-1} \leq \xi_i \leq x_i\}.$$

The binary relation, $<_r$, defined by $P <_r P'$ if P' is a refinement of P , is a direction in \mathcal{P} , and so $(\mathcal{P}, <_r)$ is a directed set. Likewise $(\mathcal{D}, <_r)$ is a directed set, where for $d = (P, \{\xi_i\}_{i=1}^n)$ and $d' = (P', \{\xi'_i\}_{i=1}^m)$, $d <_r d'$ means $P <_r P'$. Now we define the binary relation, $<_n$, by $P <_n P'$ if $\text{norm } P' \leq \text{norm } P$. This too is a direction in \mathcal{D} . If for $d = (P, \{\xi_i\}_{i=1}^n)$, we take $\text{norm } d$ to mean $\text{norm } P$ then $<_n$ is a direction in \mathcal{P} . Thus $(\mathcal{P}, <_n)$ and $(\mathcal{D}, <_n)$ are directed sets. We are now prepared to restate the definition of an integral as the limit of a net.

In this chapter our discussion is restricted to those functions which are defined and bounded on $[a,b]$.

Definition 19: The Riemann integral of f over $[a,b]$ is the number I if the net $R_f: (\mathcal{D}, <_n) \rightarrow \mathcal{R}$ defined by

$$R_f(P, \{\xi_i\}_{i=1}^n) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$$

converges to I .

Definition 20: The Darboux integral of f is I if the net

$\bar{D}_f: (\mathcal{P}, <_n) \rightarrow \mathcal{R}$ defined by

$$\bar{D}_f(P) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$

and the net $\underline{D}_f: (P, <_n) \rightarrow R$ defined by

$$\underline{D}_f(P) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$

both converge to I .

Definition 21: The Pollard integral of f is I if the net $P_f: (D, <_r) \rightarrow R$ defined by

$$P_f(P, \{\xi_i\}_{i=1}^n) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$$

converges to I .

Definition 22: The Darboux-Pollard integral of f is I if the net $\overline{DP}_f: (P, <_r) \rightarrow R$ defined by

$$\overline{DP}_f(P) = \sum_{i=1}^n M_i (x_i - x_{i-1})$$

and the net $\underline{DP}_f: (P, <_r) \rightarrow R$ defined by

$$\underline{DP}_f(P) = \sum_{i=1}^n m_i (x_i - x_{i-1})$$

both converge to I .

The equivalence of the Riemann, Darboux, Pollard and Darboux-Pollard integrals, as defined in Chapter I, is well known. In [5], Kelley sketches the outline of a proof for the equivalence of the Riemann, Darboux and Pollard integrals in the setting of convergence theory. The development in this chapter is more complete since it includes the Darboux-Pollard integral and uses the more recent idea of approximate subnets. Thus Theorem 1 is a more complete statement than Kelley makes.

Lemma 5: The net P_f is a subnet of R_f .

Proof: Let $N: (D, <_r) \rightarrow (D, <_n)$ be the identity map. Let $d \in (D, <_n)$, say $d = (P, \{\xi_i\}_{i=1}^n)$, then $d \in (D, <_r)$. Let $d_r >_r d$, say $d_r = (P', \{\xi_i\}_{i=1}^m)$. Then $P' >_r P$ which implies that

norm $P' \leq \text{norm } P$. Thus $P' \supset_n P$ so $d_r \supset_n d$, and

$$\{N(d_r) \mid d_r \supset_r d\} \subset \{d_n \mid d_n \supset_n d\}.$$

Therefore P_f is a subnet of R_f .

Lemma 6: [8] The net R_f is an approximate subnet of P_f .

Proof: Let $\epsilon > 0$, let $d_r \in (\mathcal{D}, <_r)$ say

$d_r = (P, \{\xi_i\}_{i=1}^k)$, $P = \{x_i\}_{i=0}^k$. We show that there is a $d_n \in (\mathcal{D}, <_n)$ such that

$$(1) \{R_f(d_n') \mid d_n' \supset_n d_n\} \subset N_\epsilon(\{P_f(d_r') \mid d_r' \supset_r d_r\}).$$

There exists a δ , $1 > \delta > 0$, such that $\text{norm } d_n < \delta$ implies (1) is

true. Choose $d_n \in (\mathcal{D}, <_n)$ such that $\text{norm } d_n < \delta$ for $\delta \leq \frac{\epsilon}{2MK}$

where $-M \leq f(x) \leq M$ for all $x \in [a, b]$, and K is the number of

intervals in the partition P associated with d_r . Let

$d_n' \supset_n d_n$, say $d_n' = (P', \{\zeta_i\}_{i=1}^n)$, $P' = \{y_i\}_{i=0}^n$. Form $d^* = d_n' \vee d_r$,

as suggested by Pettis in [8], $d^* = (P^*, \{\sigma_i\}_{i=1}^m)$, where

$P^* = \{y_i\}_{i=0}^n \cup \{x_i\}_{i=0}^k$. Relabel these points so that $P^* = \{z_i\}_{i=0}^m$,

m being the number of distinct intervals in P^* . Now choose

$$\{\sigma_i\}_{i=1}^m \text{ by } \sigma_i = \begin{cases} \zeta_j & \text{if there is a } \zeta_j \text{ such that } z_{i-1} \leq \zeta_j \leq z_i \\ z_i & \text{otherwise} \end{cases}$$

The partition $P = \{x_i\}_{i=0}^k$ adds at most $K - 1$ points to the

partition $P' = \{y_i\}_{i=0}^n$ to form the partition P^* . Therefore the

points $\{x_i\}_{i=0}^k$ repartition no more than $K - 1$ of the intervals

$[y_{i-1}, y_i]$. There are at most $K - 1$ non-zero terms in the difference

$|R_f(d^*) - R_f(d_n')|$ and each of these is less than $\delta \cdot 2M$ which

is equal to $\frac{\epsilon}{K}$. The other terms are all zero. Therefore the

difference is less than ϵ , that is

$$\begin{aligned} & | R_f(d^*) - R_f(d_n') | = \\ & \left| \sum_{i=1}^m f(\sigma_i) (z_i - z_{i-1}) - \sum_{i=1}^n f(\zeta_i) (y_i - y_{i-1}) \right| < \epsilon. \end{aligned}$$

To see this, look at a subinterval $[y_{i-1}, y_i]$ of $[a, b]$ from the partition P' . If there is no x_k such that $y_{i-1} < x_k < y_i$ then the term $f(\zeta_i) (y_i - y_{i-1})$ in $R_f(d_n')$ is equal to the term $f(\sigma_k) (z_k - z_{k-1})$ in $R_f(d^*)$, $z_k = y_i$, $z_{k-1} = y_{i-1}$ and $\sigma_k = \zeta_i$. This case corresponds to a zero term in the difference

$$| R_f(d^*) - R_f(d_n') |.$$

If not, then there is an x_k such that $y_{i-1} < x_k < y_i$. There could be more than one partition point from P in the interval $[y_{i-1}, y_i]$. However, the result is still true and the reasoning is similar.

Assume there is exactly one x_k such that $y_{i-1} < x_k < y_i$. Then the interval $[y_{i-1}, y_i]$ is divided into the two intervals $[y_{i-1}, x_k]$ and $[x_k, y_i]$. Relabel these $[z_{j-2}, z_{j-1}]$ and $[z_{j-1}, z_j]$. So the term $f(\zeta_i) (y_i - y_{i-1})$ in the sum $R_f(d_n')$ is relabeled $f(\zeta_i) (z_j - z_{j-2})$ and corresponds to $f(\sigma_{j-1}) (z_{j-1} - z_{j-2}) + f(\sigma_j) (z_j - z_{j-1})$. Notice that $f(\zeta_i) (z_j - z_{j-2}) = f(\zeta_i) (z_{j-1} - z_{j-2}) + f(\zeta_i) (z_j - z_{j-1})$. Now either σ_{j-1} or σ_j is equal to ζ_i (possibly both), say $\sigma_{j-1} = \zeta_i$. This case results in a non-zero term in the difference $| R_f(d_n') - R_f(d^*) |$, but still

$$\begin{aligned}
& | f(\zeta_1) (z_j - z_{j-2}) - (f(\sigma_{j-1}) (z_{j-1} - z_{j-2}) + f(\sigma_j) (z_j - z_{j-1})) | = \\
& | f(\zeta_1) (z_j - z_{j-1}) - f(\sigma_j) (z_j - z_{j-1}) | = \\
& | (f(\zeta_1) - f(\sigma_j)) (z_j - z_{j-1}) | = \\
& | f(\zeta_1) - f(\sigma_j) | \cdot | (z_j - z_{j-1}) | \leq 2M \cdot \delta = \frac{\varepsilon}{K}.
\end{aligned}$$

A non-zero difference occurs only when there is an x_k such that $y_{i-1} < x_k < y_i$. Since there are at most $K - 1$ points of the partition P different from the points of the partition P' , this can happen no more than $K - 1$ times. Therefore $| R_f(d_n') - R_f(d^*) | < \varepsilon$ and so $R_f(d_n') \in N_\varepsilon(R_f(d^*))$. But $R_f(d^*) = P_f(d^*)$, so $R_f(d_n') \subset N_\varepsilon(P_f(d^*))$. Also $d^* > \frac{1}{r} d_r$ so (1) is true. Thus R_f is an approximate subnet of P_f .

Now assume there are two points of the partition P between y_{i-1} and y_i . Then the term $f(\zeta_1) (y_i - y_{i-1})$ in the sum $R_f(d_n')$ is relabeled $f(\zeta_1) (z_j - z_{j-1})$ and corresponds to $f(\sigma_{j-2}) (z_{j-2} - z_{j-3}) + f(\sigma_{j-1}) (z_{j-1} - z_{j-2}) + f(\sigma_j) (z_j - z_{j-1})$ in the sum $R_f(d^*)$. Assume $\sigma_j = \zeta_1$. Then the difference is still less than $\frac{\varepsilon}{K}$. That is

$$\begin{aligned}
& | f(\zeta_1) (z_j - z_{j-3}) - [f(\sigma_{j-2}) (z_{j-2} - z_{j-3}) + f(\sigma_{j-1}) (z_{j-1} - z_{j-2}) + f(\sigma_j) (z_j - z_{j-1})] | = \\
& | f(\zeta_1) - f(\sigma_{j-2}) | \cdot (z_{j-2} - z_{j-3}) + | f(\zeta_1) - f(\sigma_{j-1}) | \cdot (z_{j-1} - z_{j-2}) \leq \\
& 2M(z_{j-2} - z_{j-3}) + 2M(z_{j-1} - z_{j-2}) = 2M(z_{j-1} - z_{j-3}) < 2M \cdot \delta = \frac{\varepsilon}{K}.
\end{aligned}$$

Thus no matter how many points from the partition P fall in an interval $[y_{i-1}, y_i]$, R_f is still an approximate subnet of P_f .

Lemma 7: The net \overline{DP}_f is a subnet of \overline{D}_f and \underline{DP}_f is a subnet of \underline{D}_f .

Proof: Let $N: (P, <_r) \rightarrow (P, <_n)$ be the identity map. Let $P \in (P, <_n)$, then $P \in (P, <_r)$. Let $P' >_r P$ this implies $\text{norm } P' \leq \text{norm } P$ so $P' >_n P$. Thus $\{N(P') \mid P' >_r P\} \subset \{P' \mid P' >_n P\}$. Therefore \overline{DP}_f is a subnet of \overline{D}_f , and \underline{DP}_f is a subnet of \underline{D}_f .

Lemma 8: The net \overline{D}_f is an approximate subnet of \overline{DP}_f and \underline{D}_f is an approximate subnet of \underline{DP}_f .

Proof: Similar to proof of Lemma 6.

Lemma 9: The net \overline{DP}_f is an approximate subnet of P_f and \underline{DP}_f is an approximate subnet of P_f .

Proof: Let $\varepsilon > 0$, let $d \in (D, <_r)$, say $d = (P, \{\eta_i\}_{i=1}^k)$ where $P = \{x_i\}_{i=0}^k$. Now $P \in (P, <_r)$. Let $P' >_r P$, say $P' = \{y_i\}_{i=0}^n$. Choose $d^* \in (D, <_r)$ such that $d^* = (P', \{\xi_i\}_{i=1}^n)$ where ξ_i is chosen so that $y_{i-1} \leq \xi_i \leq y_i$ and

$$M_i - f(\xi_i) < \frac{\varepsilon}{(b-a)}. \text{ Thus}$$

$$\begin{aligned} \overline{DP}_f(P') - P_f(d^*) &= \sum_{i=1}^n M_i (x_i - x_{i-1}) - \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}) = \\ &= \sum_{i=1}^n [M_i - f(\xi_i)] (x_i - x_{i-1}) \leq \sum_{i=1}^n \frac{\varepsilon}{(b-a)} (x_i - x_{i-1}) = \\ &= \frac{\varepsilon}{(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) = \frac{\varepsilon}{(b-a)} \cdot (b-a) = \varepsilon. \end{aligned}$$

Thus $\overline{DP}_f(P') \in N_\varepsilon(P_f(d^*)) \subset N_\varepsilon(\{P_f(d') \mid d' >_r d\})$, since $d^* >_r d$. Therefore \overline{DP}_f is an approximate subnet of P_f . Similarly \underline{DP}_f is an approximate subnet of P_f .

Lemma 10: The net \overline{DP}_f is a monotonically decreasing net which is bounded below and therefore converges. The net \underline{DP}_f is a monotonically increasing net which is bounded above and therefore converges.

Proof: Since f is a bounded function \overline{DP}_f and \underline{DP}_f are bounded. By the definition of M_i and m_i , \overline{DP}_f and \underline{DP}_f are monotone. Therefore by Lemma 4 they each converge.

Lemma 11: If $P \in \mathcal{P}$, say $P = \{x_i\}_{i=0}^n$, and $d = (P, \{\xi_i\}_{i=1}^n)$ for any choice of ξ_i such that $x_{i-1} \leq \xi_i \leq x_i$, then $\underline{DP}_f(P) \leq P_f(d) \leq \overline{DP}_f(P)$.

Proof: $\underline{DP}_f(P) = \sum_{i=1}^n m_i (x_i - x_{i-1})$, $P_f(d) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$ and $\overline{DP}_f(P) = \sum_{i=1}^n M_i (x_i - x_{i-1})$. Since $m_i \leq f(\xi_i) \leq M_i$ for $i = 1, \dots, n$, $\underline{DP}_f(P) \leq P_f(d) \leq \overline{DP}_f(P)$.

Lemma 12: If f is Darboux-Pollard integrable over $[a, b]$, then P_f is an approximate subnet of \overline{DP}_f and P_f is an approximate subnet of \underline{DP}_f .

Proof: Let $\varepsilon > 0$, let $P \in \mathcal{P}$. Since f is Darboux-Pollard integrable over $[a, b]$, there is $P' \in \mathcal{P}$, say $P' = \{x_i\}_{i=0}^n$, $P' >_r P$, such that $\overline{DP}_f(\hat{P}) - \underline{DP}_f(\hat{D}) < \varepsilon$ for every $\hat{P} >_r P'$. Now $d = (P', \{\xi_i\}_{i=1}^n)$ is in $(\mathcal{D}, <_r)$. Let $d^* >_r d$, say $d^* = (P^*, \{\xi_i\}_{i=1}^m)$ then $P^* >_r P'$, so $\overline{DP}_f(P^*) - P_f(d^*) < \varepsilon$ and $P_f(d^*) - \underline{DP}_f(P^*) < \varepsilon$. Therefore P_f is an approximate subnet of \overline{DP}_f and P_f is an approximate subnet of \underline{DP}_f .

Example 1: The net P_f is not always an approximate subnet of \overline{DP}_f . If the hypothesis that \overline{DP}_f and \underline{DP}_f both converge to I is omitted from Lemma 12 then Dirichlet's [2] function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

is a counterexample. It is not the case that for every $\epsilon > 0$ and for every $P \in \mathcal{P}$ there is a $d \in \mathcal{D}$ such that $\{P_f(d') \mid d' >_r d\} \subset N_\epsilon(\{\overline{DP}_f(P') \mid P' >_r P\})$. Equivalently there is an $\epsilon > 0$ and a $P \in \mathcal{P}$ such that for every $d \in \mathcal{D}$ $\{P_f(d') \mid d' >_r d\} \not\subset N_\epsilon(\{\overline{DP}_f(P') \mid P' >_r P\})$, that is there exists an $\epsilon > 0$ and a $d' >_r d$ such that there is no $P' >_r P$ for which $P_f(d') \in N_\epsilon(\overline{DP}_f(P'))$. Let $\epsilon = \frac{1}{2}$, let $P = \{0, 1\}$. Let $d \in \mathcal{D}$ say $d = (\hat{P}, \{\xi_i\}_{i=1}^n)$ where $\hat{P} = \{x_i\}_{i=0}^n$. Let $d' = (\hat{P}, \{\rho_i\}_{i=1}^n)$ where ρ_1 is an irrational number such that $x_{i-1} \leq \rho_1 \leq x_i$, then $d' >_r d$. Now $P_f(d') = 0$ but for every $P' >_r P$, $\overline{DP}_f(P') = 1$ since every interval contains a rational number. Therefore P_f is not an approximate subnet of \overline{DP}_f , since $0 \notin N_{\frac{1}{2}}(1)$. Thus the hypothesis that f be Darboux-Pollard integrable is necessary for Lemma 12.

The following theorem summarizes the results of this chapter and states what is usually said about these integrals. Lemmas 5, 6, 7, 8, 9 and 12 imply Theorem 1 and together are a more general statement about the relations among these integrals than Theorem 1.

Theorem 1: The following statements are equivalent:

- i) R_f converges to I ;
- ii) both \overline{D}_f and \underline{D}_f converge to I ;
- iii) P_f converges to I ;
- iv) both \overline{DP}_f and \underline{DP}_f converge to I .

CHAPTER IV

THE LEBESGUE CASE

The development of Lebesgue integration by Royden [11], which is equivalent to Lebesgue's own definition [6], is outlined by the following definitions. For more detail the reader may refer to [11].

Let M denote the set of Lebesgue measurable sets, and μE denote the Lebesgue measure for $E \in M$. Then for $E \in M$ we define the characteristic function of E by

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Definition 23: Let ϕ be a function defined on $E = \bigcup_{i=1}^n E_i$ by $\phi(x) = \sum_{i=1}^n c_i \chi_{E_i}$, then ϕ is a simple function if each $E_i \in M$.

Definition 24: Let ϕ be a simple function, $\{c_i\}_{i=1}^n$ be the finite set of distinct non-zero values of ϕ and let $E_i = \{x \mid \phi(x) = c_i\}$, then $\phi(x) = \sum_{i=1}^n c_i \chi_{E_i}$ is the canonical representation of ϕ . Notice that the E_i are disjoint.

Definition 25: Let ϕ be defined on $E \in M$ with μE finite, where $\phi(x) = \sum_{i=1}^n c_i \chi_{E_i}$ is the canonical representation of ϕ . Then we define the Lebesgue integral of ϕ over E by

$$L\int \phi(x) dx = \sum_{i=1}^n c_i \mu E_i.$$

For representations of ϕ which are not canonical the following lemma, also from [11], defines the integral of ϕ .

Lemma 13: Let ϕ be defined on $E = \bigcup_{i=1}^n E_i$ by

$$\phi(x) = \sum_{i=1}^n c_i \chi_{E_i},$$
 where $\{E_i\}_{i=1}^n$ is a set of disjoint measurable sets each of finite measure. Then $L\int \phi(x) dx = \sum_{i=1}^n c_i \mu E_i$.

Definition 26: Let f be a non-negative measurable function defined on $E \in M$ with μE finite. Then the Lebesgue integral of f over E is defined by

$$L\int f(x) dx = \inf L\int \psi(x) dx$$

for all simple functions $\psi \geq f$.

Definition 27: Let f be a non-negative measurable function defined on $E \in M$ then

$$L\int_E f(x) dx = \sup \int_E h(x) dx$$

where $h \leq f$ is a bounded measurable function and $\mu\{x \mid f(x) \neq 0\}$ is finite. If $L\int_E f(x) dx$ is finite then f is said to be integrable over E .

Definition 28: Let f be a measurable function. Define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Then $f = f^+ - f^-$ and f is integrable over $E \in M$ if both f^+ and f^- are integrable over E . Now we define

$$L\int_E f(x) dx = L\int_E f^+(x) dx - L\int_E f^-(x) dx.$$

Lemma 14: [11] If f is defined and bounded on $E \in M$ with μE finite then

$$\inf_{\phi \leq f} L\int_E \phi(x) dx = \sup_{\psi \geq f} L\int_E \psi(x) dx$$

for all simple functions ϕ and ψ , if and only if f is a measurable function.

Lemma 15: [11] If f is defined and bounded on $[a,b]$ then f Pollard integrable over $[a,b]$ implies f is measurable, thus integrable over $[a,b]$ and

$$L\int f(x)dx = P\int f(x)dx.$$

For this chapter all functions considered are defined and bounded on a Lebesgue measurable set E with μE finite.

The first step in redefining the Lebesgue integral in terms of convergence theory of nets is to let

$$\Pi = \{P \mid P \text{ is a partition of } E \text{ into disjoint measurable sets}\}.$$

Next let

$$\Delta = \{d = (P, \{\xi_i\}_{i=1}^n) \mid P \in \Pi, P = \{E_i\}_{i=1}^n \text{ and } \xi_i \in E_i\}.$$

Definition 29: The norm of a partition $P \in \Pi, P = \{E_i\}_{i=1}^n$ is $\max \{\mu E_i \mid E_i \in P\}$, and the norm of an element $d \in \Delta, d = (P, \{\xi_i\}_{i=1}^n)$, is $\text{norm } P$.

Definition 30: For $P, P' \in \Pi, P' = \{F_i\}_{i=1}^m$ is a refinement of $P = \{E_i\}_{i=1}^n$ if each $F_i \subset E_j$ for some $j = 1, \dots, n$.

Now let $P' >_n P, d' >_n d, P' >_r P$ and $d' >_r d$ have the same meaning as in Chapter III. Then $(\Pi, <_n)$ and $(\Delta, <_n)$ are directed sets directed by norm, while $(\Pi, <_r)$ and $(\Delta, <_r)$ are directed sets directed by refinement. We define nets on these directed sets following the style of the nets $R_f, \overline{D}_f, \underline{D}_f, P_f, \overline{DP}_f$ and \underline{DP}_f of Chapter III.

Definition 31: Let $LR_f: (\Delta, <_n) \rightarrow \mathbb{R}$ be the net defined by

$$LR_f(P, \{\xi_i\}_{i=1}^n) = \sum_{i=1}^n f(\xi_i) \mu E_i.$$

Definition 32: Let \overline{LD}_f and \underline{LD}_f be the nets defined on

$(\Pi, <_n)$ into R by

$$\overline{LD}_f(P) = \sum_{i=1}^n M_i \mu E_i \text{ and}$$

$$\underline{LD}_f(P) = \sum_{i=1}^n m_i \mu E_i, \text{ where}$$

$$M_i = \sup \{f(x) \mid x \in E_i\} \text{ and } m_i = \inf \{f(x) \mid x \in E_i\}.$$

Definition 33: Let $LP_f: (\Delta, <_r) \rightarrow R$ be the net defined by

$$LP_f(P, \{\xi_i\}_{i=1}^n) = \sum_{i=1}^n f(\xi_i) \mu E_i.$$

Definition 34: Let \overline{LDP}_f and \underline{LDP}_f be the nets defined on

$(\Pi, <_r)$ into R by

$$\overline{LDP}_f(P) = \sum_{i=1}^n M_i \mu E_i \text{ and}$$

$$\underline{LDP}_f(P) = \sum_{i=1}^n m_i \mu E_i.$$

The following integrals, $LR \int f(x) dx$, $LD \int f(x) dx$, $LP \int f(x) dx$ and $LDP \int f(x) dx$ are analogous to the Riemann, Darboux, Pollard and Darboux-Pollard integrals respectively.

Definition 35: The integral, $LR \int f(x) dx$, equals I if the net LR_f converges to I .

Definition 36: The integral, $LD \int f(x) dx$, equals I if the nets \overline{LD}_f and \underline{LD}_f both converge to I .

The directed sets $(\Delta, <_n)$ and $(\Pi, <_n)$ are so dominated by their direction, $<_n$, that these integrals are not Lebesgue integrals, despite the fact that E may be partitioned into measurable sets.

Definition 37: The integral, $LP \int f(x) dx$ equals I if the net LP_f converges to I .

Definition 38: The integral $\overline{\text{LDP}} \int f(x) dx$ equals I if both of the nets $\overline{\text{LDP}}_f$ and $\underline{\text{LDP}}_f$ converge to I .

The two integrals, $\text{LP} \int f(x) dx$ and $\text{LDP} \int f(x) dx$ are equivalent to the Lebesgue integral. To prove this, we first prove that $\text{LDP} \int f(x) dx$ is the Lebesgue integral and then that $\text{LDP} \int f(x) dx$ is equivalent to $\text{LP} \int f(x) dx$. That the two integrals $\text{LR} \int f(x) dx$ and $\text{LD} \int f(x) dx$ are not equivalent to the Lebesgue integral is shown by counterexample. In fact these integrals are no more than Riemann-type integrals, which is also proved.

Lemma 16: If $P \in \Pi$, say $P = \{E_i\}_{i=1}^n$, then $\psi(x) = \sum_{i=1}^n M_i \chi_{E_i}$ and $\phi(x) = \sum_{i=1}^n m_i \chi_{E_i}$ are simple functions defined on $E = \bigcup_{i=1}^n E_i$ such that $\psi \geq f$ and $\phi \leq f$.

Proof: There are only a finite number of M_i and m_i , $\bigcup_{i=1}^n E_i = E$ and the E_i are disjoint so ψ and ϕ are simple functions. By definition of M_i and m_i , $\psi \geq f$ and $\phi \leq f$.

Lemma 17: Let ϕ be a simple function defined on E by $\phi(x) = \sum_{i=1}^n c_i \chi_{E_i}$, then $P = \{E_i\}_{i=1}^n$ is a partition in Π .

Proof: By definition of a simple function the E_i form a finite partition of E into disjoint measurable sets. Thus $P \in \Pi$.

Lemma 18: If P and P' are in Π , and $P' >_r P$, then $\overline{\text{LDP}}_f(P') \leq \overline{\text{LDP}}_f(P)$ and $\underline{\text{LDP}}_f(P') \geq \underline{\text{LDP}}_f(P)$.

Proof: Let P and P' be in Π with $P' >_r P$, say $P = \{E_i\}_{i=1}^n$ and $P' = \{F_j\}_{j=1}^k$. Now $\overline{\text{LDP}}_f(P) = \sum_{i=1}^n M_i \mu E_i$ and $\overline{\text{LDP}}_f(P') = \sum_{j=1}^k M_j' \mu F_j$, where $F_j \subset E_i$ for some i . But $E_i = \bigcup_{j \in I_i} F_j$, so

$\overline{\text{LDP}}_f(P') = \sum_{i=1}^n \sum_{F_j \in E_1} M'_j \mu F_j \leq \sum_{i=1}^n M_i \mu E_1 = \overline{\text{LDP}}_f(P)$, since $M_i \geq M'_j$ for all j such that $F_j \in E_1$. Therefore $\overline{\text{LDP}}_f(P') \leq \overline{\text{LDP}}_f(P)$.

Similarly $m_i \leq m'_i$ for all i such that $F_i \in E_1$ so

$$\underline{\text{LDP}}_f(P') \geq \underline{\text{LDP}}_f(P).$$

Lemma 19: Let $\phi(x) = \sum_{i=1}^n c_i \chi_{E_i}$ be a simple function defined on E . Then

$$L\int \phi(x) dx = \text{LDP}\int \phi(x) dx.$$

Proof: Assume $L\int \phi(x) dx = I$. Then $I = \sum_{i=1}^n c_i \mu E_i$. Let

$P = \{E_i\}_{i=1}^n$, then

$$\overline{\text{LDP}}_f(P) = \sum_{i=1}^n M_i \mu E_i = \sum_{i=1}^n c_i \mu E_i = \sum_{i=1}^n m_i \mu E_i = \underline{\text{LDP}}_f(P). \text{ Let}$$

$P' \supset_r P$, then $\underline{\text{LDP}}_f(P) \leq \underline{\text{LDP}}_f(P') \leq \overline{\text{LDP}}_f(P') \leq \overline{\text{LDP}}_f(P)$. Thus

$$\overline{\text{LDP}}_f(P') = I = \underline{\text{LDP}}_f(P') \text{ for every } P' \supset_r P. \text{ Therefore}$$

$$\text{LDP}\int \phi(x) dx = L\int \phi(x) dx.$$

Assume $\text{LDP}\int \phi(x) dx = I$. Then both $\overline{\text{LDP}}_f$ and $\underline{\text{LDP}}_f$ converge

to I . Let $P = \{E_i\}_{i=1}^n$, then for every $P' \supset_r P$,

$$\overline{\text{LDP}}_f(P') = \sum_{i=1}^n c_i \mu E_i = \underline{\text{LDP}}_f(P'). \text{ Thus } \overline{\text{LDP}}_f \text{ and } \underline{\text{LDP}}_f \text{ both converge to } \sum_{i=1}^n c_i \mu E_i = L\int \phi(x) dx. \text{ Therefore } L\int \phi(x) dx = \text{LDP}\int \phi(x) dx.$$

Theorem 2: A bounded function f is Lebesgue integrable with $L\int f(x) dx = I$ if and only if $\text{LDP}\int f(x) dx = I$.

Proof: Assume that f is bounded and Lebesgue integrable with $L\int f(x) dx = I$. Let $\epsilon > 0$. Then

$$\sup_{\phi \leq f} L\int \phi(x) dx = \inf_{\psi \geq f} L\int \psi(x) dx = I.$$

Therefore there exists a simple function $\hat{\phi}(x) = \sum_{i=1}^n c_i \chi_{E_i}$ such

that $\hat{\phi} \leq f$ and $L\int \hat{\phi}(x) dx \in N_{\frac{\epsilon}{2}}(I)$. Also there is a simple function

$$\hat{\psi}(x) = \sum_{i=1}^m b_i \chi_{F_i} \text{ such that } \hat{\psi} \geq f \text{ and } L\int \hat{\psi}(x) dx \in N_{\frac{\epsilon}{2}}(I).$$

Then $P = \{E_i\}_{i=1}^n$ and $P' = \{F_i\}_{i=1}^m$ are in Π . Since $(\Pi, <_r)$ is a directed set there is a $\hat{P} \in \Pi$ such that $\hat{P} >_r P$ and $\hat{P} >_r P'$. Let $P^* >_r \hat{P}$, then $\overline{\text{LDP}}_f(P^*) - I \leq \overline{\text{LDP}}_f(\hat{P}) - I = L\int \hat{\psi}(x)dx - I < \epsilon$, and $I - \underline{\text{LDP}}_f(P^*) \leq I - \underline{\text{LDP}}_f(P') = I - L\int \hat{\phi}(x)dx < \epsilon$. Therefore $L\int f(x)dx = I$.

Assume $L\int f(x)dx = I$. Let $\epsilon > 0$. There exists a $P \in \Pi$, say $P = \{E_i\}_{i=1}^n$ such that $\overline{\text{LDP}}_f(P') - I < \frac{\epsilon}{2}$ and $I - \underline{\text{LDP}}_f(P') < \frac{\epsilon}{2}$ for every $P' >_r P$. Now $\hat{\psi}(x) = \sum_{i=1}^n M_i \chi_{E_i}$ and $\hat{\phi}(x) = \sum_{i=1}^n m_i \chi_{E_i}$ are simple functions such that $\hat{\phi} \leq f$ and $\hat{\psi} \geq f$. But

$$\underline{\text{LDP}}_f(P) = L\int \hat{\phi}(x)dx \leq \sup_{\phi \leq f} L\int \phi(x)dx \leq \inf_{\psi \geq f} L\int \psi(x)dx \leq L\int \hat{\psi}(x)dx = \overline{\text{LDP}}_f(P).$$

This implies that $\sup_{\phi \leq f} L\int \phi(x)dx = I = \inf_{\psi \geq f} L\int \psi(x)dx$. Therefore $L\int f(x)dx = I$.

Lemma 20: The net $\overline{\text{LDP}}_f$ is an approximate subnet of LP_f and $\underline{\text{LDP}}_f$ is an approximate subnet of LP_f .

Proof: Since $\mu E = 0$ implies both $\overline{\text{LDP}}_f$ and LP_f are constantly 0, we assume $\mu E \neq 0$. Let $\epsilon > 0$, let $d \in \Delta$, say $d = (P, \{\zeta_i\}_{i=1}^k)$ where $P = \{E_i\}_{i=1}^k$. Then $P \in \Pi$. Let $P' >_r P$, say $P' = \{E'_i\}_{i=1}^n$. Form $d^* \in \Delta$ such that $d^* >_r d$ by letting $d^* = (P', \{\xi_i\}_{i=1}^n)$ where ξ_i is chosen so that $\xi_i \in E'_i$ and $M_i - f(\xi_i) < \frac{\epsilon}{\mu E}$. Then

$$\begin{aligned} \overline{\text{LDP}}_f(P') - \text{LP}_f(d^*) &= \sum_{i=1}^n M_i \mu E'_i - \sum_{i=1}^n f(\xi_i) \mu E'_i = \\ &= \sum_{i=1}^n [M_i - f(\xi_i)] \mu E'_i \leq \sum_{i=1}^n \frac{\epsilon}{\mu E} \cdot \mu E'_i = \\ &= \frac{\epsilon}{\mu E} \cdot \sum_{i=1}^n \mu E'_i = \frac{\epsilon}{\mu E} \cdot \mu E = \epsilon. \end{aligned}$$

Therefore $\overline{\text{LDP}}_f(P') \in N_\epsilon (\text{LP}_f(d^*)) \subset N_\epsilon (\{\text{LP}_f(d') \mid d' >_r d\})$.

Similarly $\underline{\text{LDP}}_f$ is an approximate subnet of LP_f .

Lemma 21: Given $P \in \Pi$, say $P = \{E_i\}_{i=1}^k$ and any $d = (P, \{\zeta_i\}_{i=1}^k)$ then $\underline{\text{LDP}}_f(P) \leq \text{LP}_f(d) \leq \overline{\text{LDP}}_f(P)$.

Proof: Since for every i , $1 \leq i \leq n$, $m_i \leq f(\zeta_i) \leq M_i$, then $\underline{\text{LDP}}_f(P) \leq \text{LP}_f(d) \leq \overline{\text{LDP}}_f(P)$.

Lemma 22: If the $\text{LDP} \int f(x) dx$ exists then LP_f is an approximate subnet of $\overline{\text{LDP}}_f$ and LP_f is an approximate subnet of $\underline{\text{LDP}}_f$.

Proof: Let $\epsilon > 0$, let $P \in \Pi$. Since the $\text{LDP} \int f(x) dx$ exists, there is a $\hat{P} \in \Pi$ such that $\hat{P} >_r P$ and for every $\bar{P} >_r \hat{P}$, $\overline{\text{LDP}}_f(\bar{P}) - \underline{\text{LDP}}_f(\bar{P}) < \epsilon$. Choose $d \in \Delta$, say $d = (P^*, \{\xi_i\}_{i=1}^n)$ such that $P^* >_r \hat{P}$. Let $d' \in \Delta$ be such that $d' >_r d$ say $d' = (P', \{\zeta_i\}_{i=1}^k)$, then $P' >_r P^*$ so $P' >_r \hat{P}$. Therefore $\overline{\text{LDP}}_f(P') - \underline{\text{LDP}}_f(P') < \epsilon$ and so $\overline{\text{LDP}}_f(P') - \text{LP}_f(d') < \epsilon$ and $\text{LP}_f(d') - \underline{\text{LDP}}_f(P') < \epsilon$. This implies

$$\text{LP}_f(d') \in N_\epsilon (\overline{\text{LDP}}_f(P')) \subset N_\epsilon (\{\overline{\text{LDP}}_f(P') \mid P' >_r P\})$$

and

$$\text{LP}_f(d') \in N_\epsilon (\underline{\text{LDP}}_f(P')) \subset N_\epsilon (\{\underline{\text{LDP}}_f(P') \mid P' >_r P\}).$$

The function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is a counterexample to Lemma 22 if the hypothesis that $\text{LDP} \int f(x) dx$ exists is omitted. The proof of this is analogous to that for Example 1.

The integrals $LR\int f(x)dx$ and $LD\int f(x)dx$ are no more powerful than Riemann's integral. Their only advantage over Riemann's integral is that the subset of the domain of f over which f is integrated may be any measurable set. If f is defined and bounded on $[a,b]$ then the following is true:

$$LR\int f(x)dx = LD\int f(x)dx = R\int f(x)dx = D\int f(x)dx.$$

This fact is proved by the following lemmas.

Lemma 23: Let $N: (P, <_n) \rightarrow (\mathcal{D}, <_n)$ be the net defined by $N(P) = (P, \{x_i\}_{i=1}^n)$ where $P \in P$, $P = \{x_i\}_{i=0}^n$. Then $N(P)$ is cofinal in \mathcal{D} and N is order preserving.

Proof: Clearly N is order preserving. Let $d \in \mathcal{D}$, say $d = (P, \{\xi_i\}_{i=1}^n)$, then $P \in P$. Let $P' >_n P$, say $P' = \{y_i\}_{i=1}^m$. Then $N(P') = (P', \{y_i\}_{i=1}^m) = d'$ and $d' >_n d$. Therefore $N(P)$ is cofinal in \mathcal{D} .

Lemma 24: Let $N: (\mathcal{D}, <_n) \rightarrow (\Delta, <_n)$ be the net defined by $N(d) = N(P, \{\xi_i\}_{i=1}^n) = (\hat{P}, \{x_i\}_{i=1}^n)$ where $\hat{P} = \{(x_{i-1}, x_i)\}_{i=1}^n$ when $P = \{[x_{i-1}, x_i]\}_{i=1}^n$. Then $N(\mathcal{D})$ is cofinal in Δ and N is order preserving.

Proof: Clearly N is order preserving. Let $d \in \Delta$, let $\delta = \text{norm } d$. Then $d^* = (P, \{x_i\}_{i=1}^k)$ where $P = \{x_i\}_{i=0}^k$ and $a = x_0 < x_1 = x_0 + \delta < \dots < x_{k-1} = x_{k-2} + \delta < x_k = b$, is such that $d^* \in \mathcal{D}$. Now we let $d' = N(d^*)$, then $d' \in \Delta$ and $d' >_n d$. Therefore $N(\mathcal{D})$ is cofinal in Δ .

Lemma 25: Let $N: (\Delta, <_n) \rightarrow (\Pi, <_n)$ be the net defined by $N(d) = N(P, \{\xi_i\}_{i=1}^n) = P$. Then $N(\Delta)$ is cofinal in Π and N is order preserving.

Proof: Clearly N is order preserving. Let $P \in \Pi$, say $P = \{E_i\}_{i=1}^n$. Then $d = (P, \{\xi_i\}_{i=1}^n) \in \Delta$, where $\xi_i \in E_i$ for $i = 1, \dots, n$. Let $d' >_n d$, say $d' = (P', \{\xi_i\}_{i=1}^k)$ then $\text{norm } P' \leq \text{norm } P$ so $N(d') = P' >_n P$. Therefore $N(\Delta)$ is cofinal in Π .

Lemma 26: Let $N: (\Pi, <_n) \rightarrow (P, <_n)$ be the net defined by $N(P) = \hat{P}$ where $\hat{P} = \{x_i\}_{i=0}^k$,
 $a = x_0 < x_1 = x_0 + \delta < x_2 = x_1 + \delta < \dots < x_{k-1} = x_{k-2} + \delta < x_k = b$
 for $\delta = \text{norm } P$. Then N is order preserving and $N(\Pi)$ is cofinal in P .

Proof: Let $P \in P$ say $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ with $\text{norm } P = \delta$, then $P^* = \{(x_{i-1}, x_i)\}_{i=1}^n$ is in Π . Let $P' >_n P^*$, that is $\delta' = \text{norm } P' \leq \text{norm } P^* = \delta$. This implies $\text{norm } N(P') = \delta' < \delta$ so $N(P') >_n P$. Therefore $N(\Pi)$ is cofinal in P . Let $P, P' \in \Pi$ with $P' >_n P$ then $\text{norm } P' = \delta' \leq \delta = \text{norm } P$. Since $\text{norm } N(P') = \delta'$ and $\text{norm } N(P) = \delta$, $N(P') >_n N(P)$ so N is order preserving.

Theorem 3: The following statements are equivalent:

- i) R_f converges to I ;
- ii) both \bar{D}_f and D_f converge to I ;
- iii) LR_f converges to I ;
- iv) both LD_f and \underline{LD}_f converge to I .

Corollary 1: If any one of the integrals $R\int f(x)dx$, $D\int f(x)dx$, $LR\int f(x)dx$ or $LD\int f(x)dx$ exists then they all do and they are equal.

CHAPTER V

THE MCSHANE AND RIEMANN-COMPLETE INTEGRALS

In [7], Edward J. McShane defines an integral which includes, among others, the Lebesgue integral. Ralph Henstock's Riemann-complete integral also includes the Lebesgue integral. However, neither the McShane integral nor the Riemann-complete integral requires Lebesgue measure theory. Here are McShane's definitions.

Definition 39: A gauge δ defined on a set E is a neighborhood-valued function, that is it assigns $\delta(x)$ an open set containing x , to each x in E .

Definition 40: A finite set of ordered pairs, $P = \{(\bar{x}_i, A_i)\}_{i=1}^k$ is a partition of $(a,b]$, where each $A_i = (x_{i-1}, x_i]$ is a right-closed interval if $A_i \cap A_u = \emptyset$, $i \neq j$, $\bigcup_{i=1}^k A_i = (a,b]$ and $\bar{x}_i \in [a,b]$.

Definition 41: A partition, $P = \{(\bar{x}_i, A_i)\}_{i=1}^k$, is δ -fine if each $A_i \subset \delta(\bar{x}_i)$.

Definition 42: A real-valued function f is McShane integrable over $(a,b]$ if it is defined on $[a,b]$, and there is a number J such that for every $\epsilon > 0$ there is a gauge δ such that

$$P = \{(\bar{x}_i, A_i)\}_{i=1}^k \text{ a } \delta\text{-fine partition implies that}$$

$$\left| \sum_{i=1}^k f(\bar{x}_i) \ell A_i - J \right| < \epsilon,$$

where ℓA_i denotes the length of A_i . Then we say that the McShane integral of f over $(a,b]$ is J . Denote this by

$$M \int_{(a,b]} f(x) dx = J.$$

To reduce confusion between the notation of McShane's definitions and the following Riemann-complete definitions we use γ and γ -fine when referring to gauges, instead of δ and δ -fine.

Notice that $\gamma(x)$ is not necessarily an open interval, and even if it is an interval it is not necessarily centered about x . If for every x , the open set assigned to x is an open interval with x at its center, i.e. $\sigma(x) = (x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$, then σ is called a symmetric gauge. Clearly every symmetric gauge is a gauge.

As defined in [3], the Riemann-complete integral seems quite different from the McShane integral. However, Henstock redefines the Riemann-complete integral in [4]. Their similarity is then apparent. In fact they are equivalent. Here is the Riemann-complete integral as defined in [4].

Definition 43: A division of $[a, b]$ is a finite set of ordered pairs, such as $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$, where $z_i \in [x_{i-1}, x_i]$ and $a = x_0 < x_1 < x_2 < \dots < x_n = b$. The point z_i is the associated point of $[x_{i-1}, x_i]$.

Definition 44: A division $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$ is compatible with the real-valued function $\delta(x) > 0$ defined on $[a, b]$ if $|z_i - x_i| < \delta(z_i)$ and $|z_i - x_{i-1}| < \delta(z_i)$ for each i , $1 \leq i \leq n$.

Definition 45: Let f be defined in the interval $[a, b]$. If there is a number I such that for every $\epsilon > 0$ there is a function $\delta(z) > 0$ in $[a, b]$ such that $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$ compatible

with $\delta(z)$ implies

$$\left| \sum_{i=1}^n f(z_i) (x_i - x_{i-1}) - I \right| < \epsilon,$$

then f is Riemann-complete integrable and

$$\text{RC} \int_a^b f(x) dx = I.$$

Although similar, there are still some differences between the McShane and Riemann-complete definitions of the integral. A gauge γ is a neighborhood-valued function while the function δ is real-valued. McShane does not require that $\bar{x}_i \in (x_{i-1}, x_i]$ while Henstock does require that $z_i \in [x_{i-1}, x_i]$. The major difference, however, is that the gauge γ may assign to x an open neighborhood much more complicated than an open interval. Despite these differences, the two are equivalent.

We now proceed with the task of redefining both of these integrals in the setting of convergence theory of nets, and proving their equivalence.

To redefine McShane integrable in terms of convergence of a net, we first need another directed set. Let

$$G = \{(P, \gamma) \mid P \text{ is a partition, } \gamma \text{ a gauge and } P \text{ is } \gamma\text{-fine}\}.$$

We define an order on the set G by $(P', \gamma') > (P, \gamma)$ if $\gamma' \leq \gamma$, that is if $\gamma'(x) \subset \gamma(x)$ for all x in $[a, b]$.

Definition 46: Let M_f be the net defined on $(G, <) \rightarrow \mathbb{R}$ by

$$M_f(P, \gamma) = \sum_{i=1}^n f(\bar{x}_i) \ell A_i,$$

for $P = \{(x_i, A_i)\}_{i=1}^n$.

Theorem 4: The McShane integral of f over $[a, b]$ is I if and only if the net M_f converges to I .

Proof: Assume $M_f^b f(x)dx = I$. Let $\epsilon > 0$. Since $M_f^b f(x)dx = I$, there exists a gauge γ such that if $P = \{(\bar{x}_i, A_i)\}_{i=1}^n$ is γ -fine then $\left| \sum_{i=1}^n f(\bar{x}_i) \ell A_i - I \right| < \epsilon$. Let P_0 be a γ -fine partition, then $(P_0, \gamma) \in G$. Now let $(P', \gamma') > (P_0, \gamma)$, say $P' = \{(\bar{x}_i, A_i)\}_{i=1}^k$. Then P' is γ' -fine and $(P', \gamma') > (P_0, \gamma)$ so P' is γ -fine and thus $\left| \sum_{i=1}^k f(\bar{x}_i) \ell A_i - I \right| < \epsilon$. Therefore $M_f(P', \gamma') \in N_\epsilon(I)$ and so M_f converges to I .

Assume M_f converges to I . Let $\epsilon > 0$. Since M_f converges to I , there exists a $(P, \gamma) \in G$ such that for every $(P', \gamma') > (P, \gamma)$, $M_f(P', \gamma') \in N_\epsilon(I)$. Since $(P, \gamma) \in G$, γ is a gauge. Let $P' = \{(\bar{x}_i, A_i)\}_{i=1}^k$ be any γ -fine partition, then $(P', \gamma) \in G$ and $(P', \gamma) > (P, \gamma)$. Hence $M_f(P', \gamma) \in N_\epsilon(I)$. Therefore $\left| \sum_{i=1}^k f(\bar{x}_i) \ell A_i - I \right| < \epsilon$ and so $M_f^b f(x)dx = I$.

To later prove the equivalence of the McShane and Riemann-complete integrals we now introduce a modification of the McShane integral. This modification is stated in terms of nets. Let

$S = \{(P, \sigma) \mid P = \{(\bar{x}_i, A_i)\}_{i=1}^k, P \text{ a partition such that}$

$\bar{x}_i \in A_i, \sigma \text{ is a symmetric gauge and } P \text{ is } \sigma\text{-fine}\}.$

Define $(P', \sigma') > (P, \sigma)$ to mean $\sigma' \leq \sigma$, then $(S, <)$ is a directed set. Notice that $(P, \sigma) \in S$ implies $(P, \sigma) \in G$. Now we define \hat{M}_f to be the restriction of M_f to $(S, <)$.

Definition 47: Let M_f be the net defined on $(S, <) \rightarrow \mathbb{R}$

by

$$\hat{M}_f(P, \sigma) = M_f(P, \sigma).$$

Definition 48: The $\hat{M}_f^b f(x)dx = I$ if the net \hat{M}_f converges

to I .

Given a gauge γ there is an associated symmetric gauge σ_γ such that $\sigma_\gamma \leq \gamma$. We define the symmetric gauge σ_γ by setting $\sigma_\gamma(x)$ equal to the maximal symmetric interval about x such that $\sigma_\gamma(x) \subset \gamma(x)$. Thus if we prove that for every gauge γ there is a partition P_γ which is σ_γ -fine then for every gauge γ there is a P_γ which is γ -fine. Although this may seem intuitively true it requires a proof similar to that of the Heine-Borel Theorem.

Lemma 27: For every gauge γ there is a P , a partition of $(a, b]$, which is σ_γ -fine.

Proof: Let

$$S = \{x \leq b \mid \text{there is a partition of } (a, x] \text{ which is } \sigma_\gamma\text{-fine}\}.$$

As in the proof of the Heine-Borel Theorem, S is a non-empty set which is bounded above with least upper bound y and $y = b$.

Lemma 27 is the basis for the equivalence of $M \int_a^b f(x) dx$ and $\hat{M} \int_a^b f(x) dx$. The important step is the existence of a pair $(P_\gamma, \sigma_\gamma) \in S$ for every gauge γ , such that $\sigma_\gamma \leq \gamma$.

Lemma 28: Let $N: (S, <) \rightarrow (G, <)$ be the identity map. Then $N(S)$ is cofinal in G and N is order preserving.

Proof: Clearly N is order preserving. Let $(P, \gamma) \in G$, then γ is a gauge and by Lemma 27 there exists a $(P_\gamma, \sigma_\gamma) \in S$ such that $(P_\gamma, \sigma_\gamma) > (P, \gamma)$. Thus $N(P_\gamma, \sigma_\gamma) > (P, \gamma)$. Therefore $N(S)$ is cofinal in G .

Corollary 2: Then net \hat{M}_f is a subnet of M_f .

Lemma 29: Let $N: (G, <) \rightarrow (S, <)$ be the net defined by $N(P, \gamma) = (P_\gamma, \sigma_\gamma)$, where P_γ and σ_γ are constructed as in Lemma 27. Then $N(G)$ is cofinal in S and N is order preserving.

Proof: Clearly N is order preserving. Let $(P, \sigma) \in S$ then σ is a gauge and $(P, \sigma) \in G$. Thus $N(P, \sigma) = (P_\sigma, \sigma_\sigma) > (P, \sigma)$. Therefore $N(G)$ is confinal in S .

Corollary 3: The net M_f is a subnet of \hat{M}_f .

Theorem 5: The net M_f converges to I if and only if \hat{M}_f converges to I .

Proof: Lemma 1.

Corollary 4: The McShane integral is equal to the modified McShane integral.

This step, defining the modified McShane integral and proving its equivalence to the McShane integral, is an intermediate step to proving the equivalence of the McShane and Riemann-complete integrals.

Now we redefined the Riemann-complete integral in the setting of convergence theory for nets. First let

$D = \{(d, \gamma(x)) \mid d \text{ is a division and } d \text{ is compatible with } \delta(x)\}$. Then $(D, <<)$ is a directed set, where $(d, \delta(x)) << (d', \delta'(x))$ means $\delta'(x) \leq \delta(x)$ for all $x \in [a, b]$.

Definition 49: Let RC_f be the net defined on $(D, <<)$ by

$$RC_f(d, \delta(x)) = \sum_{i=1}^k f(z_i) (x_i - x_{i-1}),$$

where $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^k$.

Theorem 6: The Riemann-complete integral of f over $[a, b]$ is I if and only if the net RC_f converges to I .

Proof: The proof is similar to the proof of Theorem 4.

The following lemmas point out the natural correspondence between the real-valued functions δ of the Riemann-complete integral

and the neighborhood-valued symmetric gauges σ of the modified McShane integral. These lemmas imply that the net RC_f converges to I if and only if the net \hat{M}_f converges to I . Thus $\hat{M} \int f(x) dx = RC \int f(x) dx$ which implies $M \int f(x) dx = RC \int f(x) dx$.

Lemma 30: Let $N: (D, <) \rightarrow (S, <)$ be the net defined by $N(d, \delta(x)) = (P, \sigma)$ where $P = \{(\bar{x}_1, (x_{i-1}, x_i))\}_{i=1}^n$ when $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$, $\bar{x}_1 = z_1$ and σ is defined by $\sigma(x) = (x - \delta(x), x + \delta(x))$. Then N is order preserving and $N(D)$ is cofinal in S .

Corollary 5: The net RC_f is a subnet of \hat{M}_f .

Lemma 31: Let $N: (S, <) \rightarrow (D, <)$ be the net defined by $N(P, \sigma) = (d, \delta(x))$ where $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$ when $P = \{(\bar{x}_1, (x_{i-1}, x_i))\}_{i=1}^n$, $\bar{x}_1 = z_1$ and $\delta(x) = \delta_\sigma(x)$, the radius of $\sigma(x)$. Then N is order preserving and $N(S)$ is cofinal in D .

Proof: Clearly N is order preserving. Let $(d, \delta(x)) \in D$. Then $(P, \sigma) \in S$ where $P = \{(\bar{x}_1, (x_{i-1}, x_i))\}_{i=1}^n$ when $d = \{(z_i, [x_{i-1}, x_i])\}_{i=1}^n$, $\bar{x}_1 = z_1$, and σ is defined by $\sigma(x) = (x - \delta(x), x + \delta(x))$. Then $N(P, \sigma) = (d, \delta(x)) >> (d, \delta(x))$. Thus $N(S)$ is cofinal in D .

Corollary 6: The net \hat{M}_f is a subnet of RC_f .

Theorem 7: The net \hat{M}_f converges to I if and only if RC_f converges to I .

Proof: Lemma 1.

Corollary 7: The Riemann-complete integral is equal to the modified McShane integral.

Theorem 8: The Riemann-complete integral is equal to the McShane integral.

The McShane and Riemann-complete integrals are generalizations of the Riemann and the Lebesgue integrals. The following theorem proves that if a function is Riemann integrable then it is Riemann-complete integrable, thus also McShane integrable with all three integrals equal.

Theorem 9: If R_f converges to I then RC_f converges to I .

Proof: Let $\epsilon > 0$. There exists a $\delta > 0$ such that norm $P < \delta$ implies $|R_f(P, \{\xi_i\}_{i=1}^n) - I| < \epsilon$. Let $\delta(x) = \delta$ for all x , then d compatible with $\delta(x)$ implies

$$|RC_f(d, \delta(x)) - I| < \epsilon.$$

The following Monotone Convergence Theorem is true for the Lebesgue integral [11], the McShane integral [7] and Henstock's Riemann-complete integral [4]. We use the Monotone Convergence Theorem to prove that the McShane integral is a generalization of the Lebesgue integral. Therefore it is stated here, in general, for all three integrals.

Theorem 10: Monotone Convergence Theorem: Let $\{f_n(x)\}$ be a monotone increasing sequence of integrable functions. Then if $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ is finite and $\lim_{n \rightarrow \infty} \int f_n(x)$ is finite then $f(x)$ is integrable with

$$\int f(x) = \lim_{n \rightarrow \infty} \int f_n(x).$$

Lemma 32: Let ϕ be a simple function defined on $(a, b]$. Then M_ϕ is an approximate subnet of LP_ϕ .

Proof: Let $\phi(x) = \sum_{j=1}^n c_j \chi_{E_j}$ be a simple function bounded by M . Let $\varepsilon > 0$, let $d \in \Delta$. Now $d' = (P, \{\xi_j\}_{j=1}^n) \in \Delta$ where $P = \{E_j\}_{j=1}^n$ and $\xi_j \in E_j$. There exists a $d^* \in \Delta$ such that $d^* >_r d$ and $d^* >_r d'$, say $d^* = (\bar{P}, \{\bar{\xi}_i\}_{i=1}^m)$, $\bar{P} = \{\hat{E}_i\}_{i=1}^m$. Since each \hat{E}_i is a measurable set it can be covered by an open set O_i , which is expressible as a union of open intervals, i.e. $O_i = \bigcup_{j=1}^{\infty} I_{i,j}$, such that $\mu(O_i \setminus \hat{E}_i) < \frac{\varepsilon}{M \cdot m^2}$. Now we define a gauge σ as follows. For each $x \in E_i$, let $\sigma(x)$ be any neighborhood $N(x)$ such that

$$N(x) \subset I_{i,j}.$$

By Lemma 27 we know there exists a partition \bar{P} which is σ -fine, thus $(\bar{P}, \sigma) \in S$. Let $(P', \sigma') > (\bar{P}, \sigma)$, say $P' = \{(\bar{x}_i, A_i)\}_{i=1}^k$, then P' is σ -fine. Since $d^* >_r d'$, $\hat{E}_i \subset E_j$ for some j , and $\xi_i \in E_j$ so $f(\xi_i) = c_j$. Also for every $\bar{x}_k \in \hat{E}_i$, $f(\bar{x}_k)$ is equal to that same c_j . Associate this constant c_j with \hat{E}_i by denoting it as c_{j_i} . Therefore

$$\begin{aligned} & |LP_\phi(d^*) - M_\phi(P', \sigma')| = \\ & \left| \sum_{i=1}^m f(\xi_i) \mu \hat{E}_i - \sum_{i=1}^k f(\bar{x}_i) \ell A_i \right| = \\ & \left| \sum_{i=1}^m (c_{j_i} \mu \hat{E}_i - \sum_{\bar{x}_k \in \hat{E}_i} c_{j_i} \ell A_k) \right| \leq \\ & \sum_{i=1}^m \left| c_{j_i} (\mu \hat{E}_i - \sum_{\bar{x}_k \in \hat{E}_i} \ell A_k) \right|. \end{aligned}$$

Since for every N , $1 \leq N \leq m$,

$$\begin{aligned}
\hat{E}_N \setminus \bigcup_{\bar{x}_1 \in \hat{E}_N} A_1 &\subset \bigcup_{i=1}^N (O_i \setminus \hat{E}_i), \\
\mu(\hat{E}_N \setminus \bigcup_{\bar{x}_1 \in \hat{E}_N} A_1) &\leq \sum_{i=1}^N \mu(O_i \setminus \hat{E}_i) \leq \\
\sum_{i=1}^{m-1} \frac{\epsilon}{M \cdot m^2} &\leq \frac{\epsilon}{M \cdot m}, \text{ and so} \\
\sum_{i=1}^m \left| C_{j_i} \left(\mu E_i - \sum_{\bar{x}_k \in \hat{E}_i} \ell_{A_k} \right) \right| &\leq \\
\sum_{i=1}^m \left| C_{j_i} \right| \cdot \left| \frac{\epsilon}{M \cdot m} \right| &\leq \sum_{i=1}^m \frac{\epsilon}{m} = \epsilon.
\end{aligned}$$

Thus M_ϕ is an approximate subnet of LP_ϕ .

Corollary 8: Let ϕ be a simple function, then

$$L\phi(x)dx = M\phi(x)dx.$$

Proof: Since M_ϕ is an approximate subnet of LP_ϕ , if LP_ϕ converges to I then M_ϕ converges to I . Thus

$$M\phi(x)dx = LP\phi(x)dx, \text{ but } LP\phi(x)dx = L\phi(x)dx \text{ so}$$

$$L\phi(x)dx = M\phi(x)dx.$$

Theorem 11: If f is a bounded Lebesgue integrable function then it is McShane integrable and

$$M\int f(x)dx = L\int f(x)dx.$$

Proof: Since f is Lebesgue integrable it is the limit of an increasing sequence of simple functions ϕ_n . By the Monotone Convergence Theorem

$$\begin{aligned}
L\int f(x)dx &= \lim_{n \rightarrow \infty} L\int \phi_n(x)dx \\
&= \lim_{n \rightarrow \infty} M\int \phi_n(x)dx
\end{aligned}$$

since for every n , $L\int \phi_n(x)dx = M\int \phi_n(x)dx$. Again by the Monotone Convergence Theorem

$$\lim_{n \rightarrow \infty} M\int \phi_n(x)dx = M\int f(x)dx.$$

Therefore $L\int f(x)dx = M\int f(x)dx$.

This proof for Theorem 11 is different from the proof sketched by Henstock for an analogous theorem about the Riemann-complete integral. Since f is the limit of an increasing sequence of simple functions the proof has only a one step limiting process and thus is more straightforward than Henstock's proof with its three step limiting process.

Theorem 11 implies that the Riemann-complete integral also includes the Lebesgue integral since the Riemann-complete integral is equivalent to the McShane integral.

The Riemann-complete integral seems a more natural generalization of the Riemann integral than the McShane integral. For this reason we give the following example to show how to compute an integral using McShane's definitions.

Example 2: Let f be defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational.} \end{cases}$$

Again this is Dirichlet's classic example of a function which is not Riemann integrable but is Lebesgue integrable with $L\int_{(0,1]} f(x)dx = 0$. By actual computation we show that $M\int_{(0,1]} f(x)dx = 0$. To do this we show that for every $\epsilon > 0$ there is a gauge $\gamma > 0$ such that for every γ -fine partition $P = \{(\bar{x}_i, A_i)\}_{i=1}^k$ it is true that

$$\left| \sum_{i=1}^k f(x_i) \ell A_i - 0 \right| < \epsilon.$$

Then $M \int_{(0,1]} f(x) dx = 0 = L \int_{(0,1]} f(x) dx.$

Let $\epsilon > 0$. Let r_1, r_2, r_3, \dots be an enumeration of the rationals in $(0,1]$. Such an enumeration exists since the rationals are countable. Let N be a natural number such that $2^{-N} < \epsilon$.

Define the gauge γ by

$$\gamma(x) = \begin{cases} 2^{-N-n} & \text{if } x = r_n \\ \frac{N!}{2} & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a γ -fine partition of $(0,1]$, say $P = \{(\bar{x}_i, A_i)\}_{i=1}^m$,

where each $A_i \subset \gamma(\bar{x}_i)$ as required. The partition $P = P_1 \cup P_2$

where $P_1 = \{(\bar{x}_i, A_i) \in P \mid \bar{x}_i \text{ is rational}\}$ and

$P_2 = \{(\bar{x}_i, A_i) \in P \mid \bar{x}_i \text{ is irrational}\}$. The partitions P_1 and P_2

are disjoint, that is $\{\bigcup_{P_1} A_i\} \cap \{\bigcup_{P_2} A_i\} = \emptyset$, and

$\{\bigcup_{P_1} A_i\} \cup \{\bigcup_{P_2} A_i\} = (0,1]$. The sum,

$$\sum_{P_1} \ell A_i \leq \sum_{P_1} \gamma(\bar{x}_i) < \sum_{n=1}^{\infty} 2^{-N-n} = 2^{-N} < \epsilon.$$

Therefore $\sum_{P_2} \ell A_i > 1 - \epsilon$ since $\sum_P \ell A_i = 1$. This implies that

$$\begin{aligned} 0 &< \sum_{i=1}^k f(\bar{x}_i) \ell A_i = \sum_{P_1} f(\bar{x}_i) \ell A_i + \sum_{P_2} f(\bar{x}_i) \ell A_i = \\ &1 \cdot \sum_{P_1} \ell A_i + 0 \cdot \sum_{P_2} \ell A_i < \epsilon. \end{aligned}$$

Therefore $\left| \sum_{i=1}^k f(\bar{x}_i) \ell A_i - 0 \right| < \epsilon$, and so $M \int_{(0,1]} f(x) dx = 0$.

If a gauge γ is given by $\gamma(x) = N_{\delta}(x)$ for all x and a fixed δ then we say that γ is a uniform gauge. Every uniform gauge is a symmetric gauge but not vice-versa since δ may vary with

x for a symmetric gauge. The integral obtained using a uniform gauge in McShane's definition is equivalent to the Riemann integral. This implies that the Riemann integral is a special case of the McShane integral.

CHAPTER VI

SUMMARY

In this thesis several definitions of the integral are unified in the setting of convergence theory for nets. We prove the equivalence of the Riemann, Darboux, Pollard and Darboux-Pollard integrals. Thus we may direct the sets \mathcal{D} and \mathcal{P} either by norm or by refinement to obtain the Riemann integral. The equivalence of the integrals $LR\int f(x)dx$ and $LD\int f(x)dx$ to the Riemann integral shows that direction by norm on the sets Δ and Π does not yield the Lebesgue integral. However, direction by refinement on Δ and Π does yield the Lebesgue integral. Thus the integrals $LP\int f(x)dx$ and $LDP\int f(x)dx$ are equivalent to the Lebesgue integral. The equivalent McShane and Riemann-complete integrals are generalizations of the Lebesgue integral. Thus all of the integrals discussed in this thesis, as well as several integrals not mentioned, are included in both the McShane and Riemann-complete integrals [7], [4].

BIBLIOGRAPHY

1. Darboux, G., "Memoire Sur Les Fonctions Discontinues", Annales Sc. de l'Ecole Normale Supérieure, Vol. 4, 2nd Series, 1875.
2. Hawkins, Thomas, Lebesgue's Theory of Integration, The University of Wisconsin Press, London, 1970.
3. Henstock, Ralph, Theory of Integration, Butterworth and Co. Ltd., London, 1963.
4. Henstock, Ralph, "A Riemann-type Integral of Lebesgue Power", Canadian Journal of Mathematics, Vol. 2, No. 1, 1968.
5. Kelley, John L., General Topology, D. Van Nostrand Co. Inc., Princeton, N. J., 1955.
6. Lebesgue, Henri, "Sur Une Generalisation de l'Integral Definie", De l'Academie Des Sciences Comptes Rendus, Vol. 4, 2nd Series, 1901.
7. McShane, E. J., "A Unified Theory of Integration", The American Mathematical Monthly, Vol. 80, No. 4, 1973.
8. Pettis, B. J., "Cluster Sets of Nets", Proceedings of The American Mathematical Society, Vol. 22, No. 2, 1969.
9. Pollard, S., "The Stieltjes Integral and Its Generalisations", Quarterly Journal of Mathematics, Oxford Series 49, Vol. 192, 1920.
10. Riemann, Bernhard, "Ueber den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit", Collected Works of Bernhard Riemann, Edited by H. Weber. 2nd Edition, 1953.
11. Royden, H. L., Real Analysis, The Macmillan Co., London, 1968.